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COMPACT GROUP ACTIONS AND MAPS INTO $K(\pi, 1)$-SPACES

BY

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ABSTRACT. Let G act on an aspherical manifold M. If G is a compact Lie group acting effectively and homotopically trivially then G must be abelian. We prove a much more general form of this result, thus extending results of Donnelly and Schultz. Our method gives us a splitting result for torus actions complementing a result of Conner and Raymond. We also generalize a theorem of Schoen and Yau on homotopy equivariance.

Introduction. A closed manifold N is *aspherical* if its universal covering $\tilde{N}$ is contractible. A lot of information has been obtained by Conner and Raymond **[5–7]** regarding group actions on such manifolds. One of their results is that a connected compact Lie group acting effectively on N is abelian. On the other hand a classical theorem of Bieberbach states that any finite group (abelian or not) can act freely on some torus—the simplest examples of aspherical manifolds. Any element of a connected group acting on N defines a homeomorphism of N which is homotopic (isotopic) to the identity. Such actions are called *homotopically trivial.* We show that a finite (more generally, a compact Lie) group acting effectively and homotopically trivially on an aspherical manifold is abelian.

Actually we work with a much larger class of manifolds. Recently Donnelly and Schultz **[8]** showed that most symmetry properties of aspherical manifolds are also enjoyed by a *hyperaspherical manifold*, a closed oriented manifold equipped with a degree-1 map into an oriented aspherical manifold. We consider what we call K-manifolds, a larger class still retaining such properties. A closed oriented m-manifold M is called a *K-manifold* if there is a torsion-free group Γ together with a map f: $N \to K(\Gamma, 1)$ so that $f^*\colon H^m(\Gamma; \mathbf{Z}) \to H^m(M; \mathbf{Z})$ is onto.

The homotopy type of an aspherical manifold is determined by its fundamental group. Therefore it is natural that their symmetry properties are reflected largely in the automorphisms of the fundamental group. More accurately, when the action of a group G fixes a base point x in M there is an induced action of G on $\pi_1(M, x)$. When the base point is not necessarily fixed, this "action" is determined only up to

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conjugation in $\pi_1(M)$. That is, there is a homomorphism ψ of G into the group of outer automorphisms of $\pi_1(M)$. If M is a K-manifold and if G preserves the kernel of f_*: $\pi_1(M) \to \Gamma$, then we consider the composition of ψ with the natural map $\mathrm{Out}(\pi) \to \mathrm{Out}(\Gamma)$. The necessary algebraic data is summarized in diagram (2.1). Our main theorem in §2 states that the kernel of this homomorphism is an abelian subgroup of the compact Lie group G (Theorem 2.5). Now it immediately follows that any compact Lie group that acts effectively and homotopically trivially on a K-manifold is abelian.

The proof is based on a key lemma (Lemma 2.4) showing that a certain centralizer is torsion free and the group theoretical fact (Corollary 1.3) that a torsion-free central extension of an abelian group by a finite group is necessarily abelian.

Our method allows us to show a splitting theorem of a K-manifold: Let M be a K-manifold via f: $M \to K(\Gamma, 1)$, where Γ has homologically injective center (for example, $K(\Gamma, 1)$ = closed manifold with nonpositive sectional curvature). Then any effective torus action (T^k, M) is homologically injective. Consequently, M has a finite covering which splits into $T^k \times$ (a closed manifold).

Finally, we have a result on hyperaspherical manifolds: Let f: $M \to N$ be a degree-1 map, where N is a "nice" aspherical manifold (e.g. flat Riemannian, infranilmanifold, etc.). Let a finite group G act effectively on M. Then there is an effective G action on N for which f is homotopy equivariant if and only if G preserves $\ker f_*$: $\pi_1 M \to \pi_1 N$. This generalizes a theorem of Schoen and Yau [**16**]. We have not only deleted the smoothness condition of (G, M), but also expanded the class of N.

1. Basic group theory. In this section we shall prove that a torsion-free central extension of a group by a finite group is abelian (see Corollary 1.3 below). Actually we prove a stronger fact which is pleasant in its own right. We use nonhomogeneous cocycles to express a group extension; see [**14**, Chapter IV] for example.

LEMMA 1.1. *Let* $0 \to C \to E \to G \to 1$ *be a central extension with* G *a finite group of order* n. *Then* φ: $E \to C$, *defined by* $\varphi(x) = x^n$, *is a homomorphism.*

PROOF. Let f: $G \times G \to C$ be a 2-cocycle giving rise to the extension E. In other words, for all $\alpha, \beta, \gamma \in G$,

$$(\delta f)(\alpha, \beta, \gamma) = f(\beta, \gamma) - f(\alpha\beta, \gamma) + f(\alpha, \beta\gamma) - f(\alpha, \beta) = 0.$$

E is $C \times G$ with the group operation

$$(a, \alpha)(b, \beta) = (a + b + f(\alpha, \beta), \alpha\beta)$$

for all $a, b \in C$ and all $\alpha, \beta \in G$. We may choose f so that $f(\alpha, 1) = f(1, \alpha) = 0$. Define g: $G \to C$ by $g(\alpha) = \sum_{t \in G} f(\alpha, t)$. It is easily checked that, for all $\alpha, \beta \in G$,

$$(\delta g)(\alpha, \beta) = g(\beta) - g(\alpha\beta) + g(\alpha) = nf(\alpha, \beta).$$

If we define φ: $E \to C$ by $\varphi(a, \alpha) = na + g(\alpha)$, φ is seen to be a homomorphism. To show that $\varphi(a, \alpha) = (a, \alpha)^n$ we see that, by computation,

$$(a, \alpha)^n = \left(na + \sum_{i=1}^{n} f(\alpha, \alpha^i), 1\right).$$

Hence we will be done when we demonstrate $g(\alpha) = \sum_{i=1}^{n} f(\alpha, \alpha^i)$.

Putting $\alpha^i = \beta$ in $\delta f \equiv 0$ gives, for all $\gamma \in G$,

$$f(\alpha, \alpha^i) - f(\alpha, \alpha^i\gamma) = f(\alpha^i, \gamma) - f(\alpha^{i+1}, \gamma).$$

Summing over $1 \leqslant i \leqslant m$ = order of α, the right-hand side disappears, i.e., for all $\gamma \in G$,

$$\sum_{i=1}^{m} f(\alpha, \alpha^i) = \sum_{i=1}^{m} f(\alpha, \alpha^i\gamma).$$

When γ ranges over coset representatives for the subgroup generated by α in G, any $t \in G$ can be expressed uniquely as $t = \alpha^i\gamma$ $(1 \leqslant i \leqslant m)$. Therefore

$$\sum_{t \in G} f(\alpha, t) = \sum_{\gamma} \sum_{i=1}^{m} f(\alpha, \alpha^i\gamma) = \sum_{i=1}^{n} f(\alpha, \alpha^i). \quad \square$$

The next corollary was shown, by a geometric argument, in [**11**, Fact 2] when C is finitely generated and torsion free (i.e., $C \cong \mathbf{Z}^k$).

COROLLARY 1.2. *Let* $0 \to C \to E \to G \to 1$ *be a central extension with* G *finite. Then* $t(E)$*, the elements of finite order in* E*, form a fully invariant subgroup. Moreover,* $t(E)$ *is finite when* $t(C)$ *is finite.*

PROOF. Any endomorphism of E maps $t(E)$ into itself, so $t(E)$ is fully invariant. Since $t(E) = \varphi^{-1}(t(C))$ (φ as in Lemma 1.1), $t(E)$ is a subgroup. The quotient $t(E)/t(C)$ is finite. Therefore, $t(E)$ is finite when $t(C)$ is. $\square$

The following corollary will be crucial in our future arguments. It is a trivial consequence of Lemma 1.1. It was already known [**11**] when $C \cong \mathbf{Z}^k$.

COROLLARY 1.3. *Let* $0 \to C \to E \to G \to 1$ *be a central extension with* E *torsion free and* G *finite. Then* E *and hence* G *is abelian.*

PROOF. When E is torsion free, $\varphi(x) = x^n$ of Lemma 1.1 is injective. Hence E is isomorphic to a subgroup of the abelian group C. $\square$

REMARKS. If $0 \to C \to E \to G \to 1$ is a central extension, the Lyndon spectral sequence (with C coefficients) gives the exact sequence [**14**, p. 354, (10.6)]

$$0 \to \mathrm{Hom}(G, C) \to \mathrm{Hom}(E, C) \to \mathrm{Hom}(C, C) \overset{\delta}{\to} H^2(G; C).$$

The boundary map δ takes the identity map $1_C \in \mathrm{Hom}(C, C)$ to $[E]$, the class representing the extension E. If $m[E] = 0$, then there is a $\varphi \in \mathrm{Hom}(E, C)$ such that $\varphi|_C = m$. When G is torsion and C is torsion free, $\mathrm{Hom}(G, C) = 0$; hence, φ is unique. It is not true in general that $\varphi(x) = x^m$ for all $x \in E$. (If $E \cong C \times G$ we can take $m = 1$ and clearly that does not work. Even when $x^m \in C$ for all $x \in E$, $x \to x^m$ may not be a homomorphism. Take $E = \{\pm 1, \pm i, \pm j, \pm k\}$, the quaternions, $C = \{\pm 1\}$, and $m = 2$.) When G is finite of order n, $n[E] = 0$. If C is torsion free, $\varphi(x)^n = \varphi(x^n) = (x^n)^n$ (x^n and $\varphi(x)$ both lie in the torsion-free abelian group C) implies $\varphi(x) = x^n$. This gives an alternate proof of Corollary 1.3.

In the next section, whenever we claim that a compact Lie group is abelian we will give a proof only for a finite group. The following fact makes this possible. It should be well known to experts, but we could not locate a reference, so we provide an argument.

FACT 1.4. *If all the finite subgroups of a compact Lie group G are abelian then G is abelian.*

PROOF. First we show that G_0, the connected component of G, is a torus. Let T^r be a maximal torus (of rank r), $N(T^r)$ its nomalizer in G_0 and W the Weyl group with $n = |W|$. For any natural number k, $(\mathbf{Z}_{kn})^r = \{t \in T^r\colon t^{kn} = 1\}$ is a characteristic subgroup of T^r. Hence the standard action of W on T^r restricts to an action on $(\mathbf{Z}_{kn})^r$, giving the exact sequence of W-modules

$$0 \to (\mathbf{Z}_{kn})^r \to T^r \overset{kn}{\to} T^r \to 0.$$

In cohomology, we have the exact sequence

$$H^2\big(W; (\mathbf{Z}_{kn})^r\big) \overset{i_*}{\to} H^2(W; T^r) \overset{kn}{\to} H^2(W; T^r).$$

Now $H^2(W; -)$ is annihilated by $n = |W|$, so i_* is onto. Let $i_*[W_k] = [N(T^r)]$; where $[W_k]$ is in $H^2(W; (\mathbf{Z}_{kn})^r)$. Then

$$\begin{array}{ccccccccc} 0 & \to & (\mathbf{Z}_{kn})^r & \to & W_k & \to & W & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 1 & \to & T^r & \to & N(T^r) & \to & W & \to & 1 \end{array}$$

is a commutative diagram. Since W_k is isomorphic to a finite subgroup of G, it is abelian. Then W acts trivially on $(\mathbf{Z}_{kn})^r$ (the action is induced by conjugation in W_k). But $\bigcup_k(\mathbf{Z}_{kn})^r$ is dense in T^r, hence W acts trivially on T^r. The action of the Weyl group is always effective, so W must be trivial. Therefore $G_0 = T^r$ (see, e.g., [**3**, Proposition 20.2]).

Next consider the exact sequence

$$1 \to G_0 = T^r \to G \to F \to 1,$$

where F is finite, say of order m. Choose $[F_k]$ in $H^2(F; (\mathbf{Z}_{km})^r)$ as above, i.e., $[F_k]$ is the preimage of $[G] \in H^2(F; T^r)$. Again F_k is abelian since it is isomorphic to a finite subgroup of G, and G is abelian because $\bigcup_k F_k$ is dense in G. □

Note that a compact abelian Lie group G is isomorphic to a direct product of a torus (the connected component of the identity) with a finite abelian group. That is, the exact sequence $1 \to T \to G \to F \to 1$ splits. To get the isomorphism F into G one modifies a preimage of a generator α of F using the fact that T is a divisible group.

2. Group actions on K-manifolds. In this section we prove our main results on homotopically trivial group actions and derive some applications. We prove these results for a class of manifolds called K-manifolds. This concept generalizes the notion of hyperaspherical (a fortiori, aspherical) manifold; our results imply that many of the basic results concerning group actions on aspherical and hyperaspherical manifolds are still valid for K-manifolds.

Let M be a connected manifold and let G be a subgroup of the group of homeomorphisms of M. If $p\colon \tilde{M} \to M$ is a universal covering, we shall denote by G^* all homeomorphisms of $\tilde{M}$ covering those in G. The group G^* fits into the exact sequence

$$1 \to \pi_1(M) \to G^* \xrightarrow{\rho} G \to 1.$$

If G has a fixed point m in M, then the action of G can be lifted to $\tilde{M}$, i.e., there is a splitting $\sigma\colon G \to G^*$ and G^* is a semidirect product (see [**4**, Chapter I, §9; **7**, **15**] for details). In this case there is an induced action of G on $\pi_1(M, m)$. Let $\pi = \pi_1(M)$ and $\mathrm{Out}(\pi)$ be the group of outer automorphisms of π. In general we have the homomorphism ψ of G into $\mathrm{Out}(\pi)$. For a normal subgroup Λ of π let $\hat{M} = \tilde{M}/\Lambda$ be the corresponding cover of M. If $\hat{G}$ consists of all homeomorphism of $\hat{M}$ covering those in G, the map $\hat{G} \to G$ may not be onto. When the subgroup Λ is normal in G^*, however, $\hat{G} \cong G^*/\Lambda$ and hence $\hat{G} \to G$ is onto. Conjugation in G^* gives a homomorphism from G^* into $\mathrm{Aut}(\pi, \Lambda)$, the automorphisms of π leaving Λ invariant. There are natural homomorphisms from $\mathrm{Aut}(\pi, \Lambda)$ into $\mathrm{Aut}(\pi/\Lambda)$ and from $\mathrm{Out}(\pi, \Lambda)$ (the image of $\mathrm{Aut}(\pi, \Lambda)$ in $\mathrm{Out}(\pi)$) into $\mathrm{Out}(\pi/\Lambda)$. We have an exact commutative diagram

$$\begin{array}{ccccccccc}
1 & \to & C(\pi/\Lambda) & \to & C_{G^*/\Lambda}(\pi/\Lambda) & \to & \mathrm{Ker}\,\psi' & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \pi/\Lambda & \to & G^*/\Lambda & \to & G & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \psi' & & \\
1 & \to & \mathrm{Inn}(\pi/\Lambda) & \to & \mathrm{Aut}(\pi/\Lambda) & \to & \mathrm{Out}(\pi/\Lambda) & \to & 1
\end{array} \tag{2.1}$$

where $C(\pi/\Lambda)$ is the center of π/Λ, $C_{G^*/\Lambda}(\pi/\Lambda)$ is the centralizer of (the subgroup) π/Λ in G^*/Λ, and ψ' is the composition $G \to \mathrm{Out}(\pi, \Lambda) \to \mathrm{Out}(\pi/\Lambda)$. The exactness of the top row is a standard diagram chase. In the proof of the following theorem we will need this in the special case $\Lambda = \{1\}$.

THEOREM 2.2. *Let M be a manifold having the homotopy type of a finite $K(\pi, 1)$. If a compact Lie group G acts freely and homotopically trivially on M, then G is abelian.*

The editor has kindly brought to our attention that this result was known to Frank Raymond, and it is contained in a handwritten manuscript, which is a joint work with Walter Neumann.

PROOF. It suffices to consider finite G by Fact 1.4. Since the order of a torsion element in $G^* \cong \pi_1(\tilde{M}/G^*)$ must divide $1 = \chi(\tilde{M})$ [**9**, Corollary 3], G^* is torsion free. As G acts homotopically trivially, $G = \mathrm{Ker}\,\psi$. In the central extension

$$1 \to C(\pi) \to C_{G^*}(\pi) \to G \to 1,$$

$C_{G^*}(\pi)$ is torsion free, hence by Corollary 1.3, G is abelian. □

The assumptions of freeness and homotopic triviality of the action are necessary because any finite group can act freely on some torus and any compact Lie group can act effectively (and homotopically trivially) on some finite-dimensional representation space. When we consider a closed manifold M we can relax the hypothesis quite a bit. We need the following

DEFINITIONS 2.3. We shall say that an action (of G on M) is *inner on* π if ψ: $G \to \mathrm{Out}(\pi)$ is trivial ($\pi = \pi_1(M)$). In particular, homotopically trivial actions are inner, e.g., any action of a connected group G. An action is said to *preserve a normal subgroup* Λ *of* π if Λ is normal in G^*. An inner action will preserve any normal subgroup and a characteristic subgroup will be preserved under any action. We say that an action is *inner on* π/Λ if the action preserves Λ and the homomorphism ψ': $G \to \mathrm{Out}(\pi/\Lambda)$ is trivial.

We shall call a closed oriented manifold M a *K-manifold* if there is a torsion-free group Γ and a map f: $M \to K(\Gamma, 1)$ such that f^*: $H^m(K(\Gamma,1); \mathbf{Z}) \to H^m(M; \mathbf{Z})$ is onto, where $m = \dim M$. We may assume that f_*: $\pi \to \Gamma$ is onto since f factors (up to homotopy) through $K(f_*(\pi), 1)$. The hyperaspherical manifolds of [**8**] are K-manifolds with $K(\Gamma, 1)$ a closed oriented manifold and f a degree-1 map.

The property of being a K-manifold is intrinsic, i.e., independent of f and Γ. Let $t(\pi)$ be the smallest normal subgroup of π containing all torsion. Define $t^{k+1}(\pi) = p_k^{-1}(t(\pi/t^k(\pi)))$ inductively, where p_k: $\pi \to \pi/t^k(\pi)$ is the natural projection. Let $t^\infty(\pi) = \bigcup_k t^k(\pi)$. Then $\pi/t^\infty(\pi)$ is torsion free, and any homomorphism $\pi \to \Gamma$ (Γ torsion free) factors through $\pi/t^\infty(\pi)$. Thus M is a K-manifold if and only if

$$H^m(\pi/t^\infty(\pi); \mathbf{Z}) \to H^m(M; \mathbf{Z})$$

is onto. Since $t^\infty(\pi)$ is a characteristic subgroup of π, any G action preserves $t^\infty(\pi)$.

The next lemma will be our main tool for most of our results. It is a generalization of [**11**, Lemma 1]. Our proof is based on ideas of [**8**].

LEMMA 2.4. *If G acts on a K-manifold M preserving* $\Lambda = \mathrm{Ker} f_*$, *then* $C_{G^*/\Lambda}(\pi/\Lambda)$ *is torsion free.*

PROOF. Assume $C_{G^*/\Lambda}(\pi/\Lambda)$ has torsion. Then there is an element α of prime order q. Let H be the cyclic subgroup of G generated by the image of α (H is nontrivial because $\Gamma = \pi/\Lambda$ is torsion free). Hence $1 \to \Gamma \to H^*/\Lambda \leftrightarrows H \to 1$ splits. Since α commutes with Γ, $H^*/\Lambda \cong \Gamma \times H$. So we can factor the map $\pi \to \Gamma$ as $\pi \to H^* \to H^*/\Lambda = \Gamma \times H \to \Gamma$. The covering map p: $\tilde{M} \to M$ induces a homeomorphism $\tilde{M}/H^* \to M/H$. Now, by a theorem of Armstrong [**1**], $\pi_1(\tilde{M}/H^*) \cong H^*/N$, where N is the smallest normal subgroup generated by the isotropy subgroups $\{H_x^*\}$, $x \in \tilde{M}$. It is easily checked that p: $H^* \to H$ induces isomorphisms $H_x^* \cong H_{p(x)}$ for all $x \in \tilde{M}$. Hence H_x^* are all finite. Then $H^* \to \Gamma$ factors through $H^*/N \cong \pi_1(M/H)$. The commutative triangle of groups

$$\begin{array}{ccc} \pi & \to & \pi_1(M/H) \\ f_*\downarrow & \swarrow & \\ \Gamma & & \end{array}$$

gives rise to a homotopy commutative triangle of spaces

$$\begin{array}{ccc} M & \to & M/H \\ f\downarrow & \swarrow & \\ K(\Gamma,1) & & \end{array}$$

because M/H has the homotopy type of a finite complex [**8**, Theorem 1.2]. Therefore,

$$\begin{array}{ccc} H^m(M;\mathbf{Z}) & \leftarrow & H^m(M/H;\mathbf{Z}) \\ f^*\uparrow & \nearrow & \\ H^m(K(\Gamma,1);\mathbf{Z}) & & \end{array}$$

is commutative. But $H^m(M/H;\ \mathbf{Z}) \to H^m(M;\ \mathbf{Z})$ is not onto [**8**, Lemma 2.5], contradicting the fact that f^* is onto (by the definition of a K-manifold). □

THEOREM 2.5. *Let M be a K-manifold and let G be a compact Lie group acting effectively on M with an inner action on Γ. Then G is abelian.*

PROOF. Again it suffices to consider finite G by Fact 1.4. Since the action is inner on Γ, ψ' is trivial by definition, hence $G = \operatorname{Ker}\psi'$. The top row of (2.1), $1 \to C(\Gamma) \to C_{G^*/\Lambda}(\Gamma) \to G \to 1$ is a central extension with $C_{G^*/\Lambda}(\Gamma)$ torsion free (Lemma 2.4) and G finite. Therefore G is abelian by Corollary 1.3. □

COROLLARY 2.6. *Only an abelian compact Lie group can act effectively and homotopically trivially on a hyperaspherical manifold.* □

The following is also an immediate consequence of Lemma 2.4 in conjunction with (2.1). It is due to Borel, Conner and Raymond [7] for aspherical M and to Donnelly and Schultz [**8**] for hyperaspherical M.

PROPOSITION 2.7. *Let G be a finite group acting effectively on a K-manifold M.*

(i) *If the action of G lifts to an action on $\tilde{M}$ (e.g., when G has a fixed point), then $G \to \operatorname{Aut}(\pi)$ is injective. If G preserves Λ, then $G \to \operatorname{Aut}(\pi) \to \operatorname{Aut}(\pi/\Lambda)$ is injective.*

(ii) *If π has trivial center, then $\psi\colon G \to \operatorname{Out}(\pi)$ is injective. If G preserves Λ and Γ has no center, then $\psi'\colon G \to \operatorname{Out}(\Gamma)$ is injective.*

PROOF. In either case if Λ is not given we can take it to be $t^\infty(\pi)$. For (i), $G \to \operatorname{Aut}(\Gamma)$ is injective because the kernel lies in the torsion-free group $C_{G^*/\Lambda}(\Gamma)$. To see (ii) observe that the inclusion $\ker\psi$ or $\ker\psi'$ in G factors through the torsion-free $C_{G^*/\Lambda}(\Gamma)$ ((2.1) and Lemma 2.4). □

For a K-manifold M, if Γ can be chosen to be abelian, then M is *hypertoral* [**17**]. Then $f_*\colon \pi \to \Gamma$ factors through the abelianization of π, $H_1(M;\ \mathbf{Z})$. Also, since Γ is torsion free, f_* factors through the free part of $H_1(M;\ \mathbf{Z})$. This is the universal Γ corresponding to $\pi/t^\infty(\pi)$ in the nonabelian case. Such M can also be characterized by the fact that the fundamental class in cohomology is a product of 1-dimensional classes or by admitting a degree-1 map to a torus. In particular, oriented surfaces with genus $g \geqslant 1$ are K-manifolds with $\Gamma \cong \mathbf{Z}^{2g}$. When Γ is the free part of

$H_1(M; \mathbf{Z})$, the induced G action on Γ is equivalent to the homology action of G (naturality of $\pi_1(M) \to H_1(M; \mathbf{Z})$). We can talk about the induced action on Γ, for abelian Γ, because $\mathrm{Aut}(\Gamma) \cong \mathrm{Out}(\Gamma)$. The action of G on $H_1(M; \mathbf{Q})$ is completely determined by the action of G on the free part of $H_1(M; \mathbf{Z})$. We have the following [**17**, 2.2, 2.3]

COROLLARY 2.8. *Let G be a finite group acting effectively on the hypertoral manifold M. Assume either*

(i) *the Euler-Poincaré characteristic $\chi(M)$ is relatively prime to the order of G, or*

(ii) *the cohomology algebra $H^*(M; \mathbf{Q})$ is generated by $H^1(M; \mathbf{Q})$ and the Euler-Poincaré characteristic $\chi(M)$ is nonzero.*

Then the action of G on $H_1(M; \mathbf{Q})$ is faithful.

PROOF. The argument is based on showing any element α of G having prime order q has a fixed point, and then using Proposition 2.7(i). The action of α on M cannot be free in (i) because $p \mid |G|$ implies $p \nmid \chi(M)$. In (ii), if the action of α on $H_1(M; \mathbf{Q})$ is trivial then the action on $H^*(M; \mathbf{Q})$ is trivial. Hence the Lefschetz number $\Lambda(\alpha) = \chi(M) \neq 0$ and α has a fixed point by the Lefschetz fixed point theorem. Now, by Proposition 2.7, α acts nontrivially on Γ, which can be chosen to be the free part of $H_1(M; \mathbf{Z})$. Therefore the action on $H_1(M; \mathbf{Q})$ is faithful. □

REMARKS. The conclusion of Corollary 2.8 can also be stated as: Let ψ be a homeomorphism of finite order of M. If ψ induces identity on the first homology then ψ is identity. When this holds for $H_1(M; \mathbf{Q})$ it also holds for $H_1(M; \mathbf{Z}_n)$, $n > 2$. This follows from a slight generalization of a theorem of Minkowski: The natural homomorphism $\mathrm{GL}(m, \mathbf{Z}) \to \mathrm{GL}(m, \mathbf{Z}_n)$ has torsion-free kernel for $n > 2$ (the proof is elementary). When M is a closed oriented surface of genus greater than one the hypothesis of Corollary 2.8 (ii) is satisfied. This is a classical result for conformal automorphisms of a Riemann surface. It is due to Hurwitz for rational coefficients and to Serre for $\mathbf{Z}_n$-coefficients, $n > 2$ (see [**18**, 14.5]).

The following was known for finite groups and tori [**17**].

COROLLARY 2.9. *Let G be a compact Lie group acting effectively on the hypertoral manifold M. If the action of G on $H_1(M; \mathbf{Q})$ is trivial, then G must act freely on M.*

PROOF. If G acts trivially on $H_1(M; \mathbf{Q})$, then it acts trivially on the free part of $H_1(M; \mathbf{Z})$. Let us choose Γ to be the free part of $H_1(M: \mathbf{Z})$. Since Γ is abelian and the action is inner on Γ, the top and the middle rows of (2.1) are identical. Hence G^*/Λ is $C_{G^*/\Lambda}(\Gamma)$. If there was a nontrivial isotropy group G_x, we could lift its action to $\tilde{M}$. That would mean that G^*/Λ has a nontrivial torsion, contradicting Lemma 2.4. □

3. Splitting via a torus action, homotopy equivariant actions. We prove two facts using the previous results. Firstly, any effective torus action on a K-manifold over Γ which has a homologically injective center is homologically injective. This immediately implies a splitting theorem. Secondly, let $f\colon M \to N$ be a degree-1 map where N is a "nice" manifold (see Example 3.4). Then any effective finite action on M induces an action on N, so that f is homotopy equivariant, provided that f satisfies a necessary condition. This is a topological version of [**16**, Theorem 11].

Let G be a *connected* compact Lie group acting effectively on a manifold M. Picking a base point m in M, consider the evaluation (orbit) map ev: $G \to M$ given by $\mathrm{ev}(t) = tm$ for all $t \in G$. It is well known that $\mathrm{im}(\mathrm{ev}_*)$ is a central subgroup of $\pi_1(M, m)$ [**5**]. The action of G can be lifted to a covering $\hat{M} = \tilde{M}/\Lambda$ for some subgroup Λ of $\pi_1(M, m)$, if and only if $\mathrm{im}(\mathrm{ev}_*) \subset \Lambda$ [**5**, §4]. Since G is connected, the action is homotopically trivial. If M is a K-manifold then, by Theorem 2.5, G is abelian, hence a torus. The action of G on M is *injective* (see [**5**] for aspherical M and [**8**] for rationally hyperaspherical M); that is,

PROPOSITION 3.1. *Let T be a torus acting effectively on a K-manifold M. Then the composition $f_*\mathrm{ev}_*$: $\pi_1(T) \to \pi \to \Gamma$ is injective.*

PROOF. Since any nonzero element in $\pi_1(T) \cong \mathbf{Z}^k$ can be represented by a monomorphism $S^1 \to T$, we need only consider an effective circle action. Let us obtain a contradiction by assuming $\pi_1(S^1) \to \Gamma$ is trivial. If $\Lambda = \ker f_*$, then this S^1 action can be lifted to an action on $\hat{M} = \tilde{M}/\Lambda$ by the paragraph above. Now the usual topology of S^1 is the same as the one coming from the compact-open topology. Hence the kernel of the induced homomorphism $S^1 \to \mathrm{Aut}(\Gamma)$ (see (2.1)) is closed in S^1. Since the action is homotopically trivial, this map actually factors through $\mathrm{Inn}(\Gamma)$. Being a homomorphic image of the finitely generated group $\pi_1(M)$, $\mathrm{Inn}(\Gamma)$ is countable. Hence $S^1 \to \mathrm{Inn}(\Gamma)$ is trivial. Then $S^1 < C_{G^*/\Lambda}(\Gamma)$, contradicting Lemma 2.4. □

The inclusion of a maximal torus into a nonabelian compact connected Lie group never induces a monomorphism at the fundamental group level. Hence Proposition 3.1 gives an alternate argument for the fact that only tori can act effectively on K-manifolds. Another implication is that all the isotropy subgroups must be finite, since otherwise $\pi_1(T) \to \pi_1(M)$ would not be injective.

A central subgroup of π is said to be *homologically injective* if it maps injectively into $H_1(\pi; \mathbf{Z})$ via $\pi \to H_1(\pi; \mathbf{Z})$. For example the fundamental group of a closed smooth manifold with nonpositive sectional curvature has a homologically injective center by a theorem of Wolf. A torus action is *homologically injective* if $\mathrm{im}(\mathrm{ev}_*)$ (central in π) is homologically injective. Homological injectiveness of an action has a very strong implication as shown in the following result of Conner and Raymond.

PROPOSITION 3.2 [**6**]. *Let M be a closed manifold with an effective torus action. This action is homologically injective if and only if M has a finite covering $T \times M'$ for some closed manifold M'. Furthermore, the deck transformation group is (finite) abelian, acting diagonally and freely as translations on the first factor.*

A torus action on a K-manifold M with Γ abelian is always homologically injective. Then the action is free and M splits as a product $M/T \times T$ [**17**]. This does not hold for all K-manifolds, not even for aspherical manifolds, but we have

THEOREM 3.3. *Let M be a K-manifold, where Γ has homologically injective center. Then any T (torus) action on M is homologically injective. Consequently $M = T \times M'/\Delta$, where M' is a closed manifold and Δ is a finite abelian group acting diagonally and freely as translations on T.*

PROOF. Since $\pi_1(T) \xrightarrow{\mathrm{ev}_*} \pi_1(M) \xrightarrow{f_*} \Gamma$ is a monomorphism into the center of Γ and the center of Γ is homologically injective, the commutative diagram

$$\begin{array}{ccccc} \pi_1(T) & \to & \pi_1(M) & \to & H_1(M;\mathbf{Z}) \\ & & f_*\downarrow & & \downarrow \\ & & \Gamma & \to & H_1(\Gamma;\mathbf{Z}) \end{array}$$

implies $\pi_1(T) \to \pi_1(M) \to H_1(M,\mathbf{Z})$ is injective. □

We will now consider a finite group acting effectively on a hyperaspherical manifold M. Thus we have a degree-1 map $f\colon M \to N = K(\Gamma,1)$. Can we find an effective G action on N such that f is a G-map? This should not be possible for arbitrary f. If we only require f to be *homotopically equivariant*, i.e., for any g in G,

$$\begin{array}{ccc} M & \xrightarrow{g} & M \\ f\downarrow & & f\downarrow \\ N & \to & N \end{array}$$

is homotopy commutative, then the answer is affirmative in many instances (possibly all). When the action is smooth and N is a flat Riemannian manifold, this was known [**16**, Theorem 13; **12**, Theorem 2.1] under the necessary condition that G preserves $\Lambda = \operatorname{Ker} f_*$. We will remove the smoothness condition and generalize this to a larger class of N.

When the action of G preserves $\Lambda = \operatorname{Ker} f_*$, we have the *abstract kernel* [**14**, p. 124] $\psi'\colon G \to \operatorname{Out}(\Gamma)$, where $\Gamma = \pi_1(N)$. An extension $1 \to \Gamma \to E \to G \to 1$ is called *admissible* if $C_E(\Gamma)$ (the centralizer of Γ in E) is torsion free. A necessary algebraic requirement for ψ' to admit an effective *geometric realization* (an effective G action on N inducing ψ') is the existence of an admissible extension [**11**]. Let us say that a closed aspherical manifold N is "nice" if the following holds: A finite abstract kernel $\psi'\colon G \to \operatorname{Out}(\pi_1(N))$ can be realized geometrically if and only if ψ' admits an admissible extension. No aspherical manifold which is not "nice" is known at present time.

EXAMPLES 3.4. The "model" $K(\Gamma,1)$-manifolds constructed in [**13**] for torsion-free virtually poly-$\mathbf{Z}$ groups Γ are "nice" [**13**, Theorem 6.4]. This contains all compact flat Riemannian manifolds and more generally all infranilmanifolds (almost flat manifolds). All classical Seifert manifolds in dimension 3, Seifert fiber spaces over a locally symmetric space are "nice". More generally all Seifert fiber spaces over "nice" manifolds are "nice" [**13**]. All hyperbolic manifolds are "nice" by Mostow's Rigidity theorem in dimension $\geqslant 3$, and by [**10**] in dimension 2.

THEOREM 3.5. *Let M be a hyperaspherical manifold with $f\colon M \to N$, where N is "nice". Let a finite group G act effectively on M. There is an effective G action on N for which f is homotopy equivariant if and only if G preserves* $\operatorname{Ker} f_*$.

PROOF. By Lemma 2.4, $1 \to \Gamma \to G^*/\Lambda \to G \to 1$ is admissible ($\Gamma = \pi_1(N)$, $\Lambda = \operatorname{Ker} f_*$). Since N is "nice", this extension can be geometrically realized. The commutativity of

$$\begin{array}{ccccccccc} 1 & \to & \pi_1(M) & \to & G^* & \to & G & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow = & & \\ 1 & \to & \pi_1(N) & \to & G^*/\Lambda & \to & G & \to & 1 \end{array}$$

guarantees the homotopy equivariance of f. □

ADDED IN PROOF. Examples of K-manifolds which are not hyperaspherical will appear in a paper by K. B. Lee and F. Raymond entitled *Manifolds on which only tori can act*.

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